Gessel-Type Expansion for the Circular β-Ensemble and Central Limit Theorem for the Sine-β process for $\beta \leq 2$

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The sine-β process was first introduced by Killip, Stoiciu [2] and Valkó, Virág [1] as the microscopic limit of the circular and Gaussian β-ensembles. It is a random countable discrete set of particles on the real that, due to the result of Dereudre, Hardy, Leblé, Maida [9], may be viewed as a Gibbs measure with the pair logarithmic potential, the parameter $\beta > 0$ being the inverse temperature.

For $\beta = 2$, this process may be explicitly described by its correlation functions, which admit an expression in terms of determinants of the sine kernel. In this case Soshnikov [8] established that the sum $\sum_{x \in X} f(x/R)$ of a sufficiently smooth function f at the points of a random subset $X \subset \mathbb{R}$ approaches Gaussian distribution as $R \to \infty$. We are able to show that this convergence holds in the entire large-temperature range $\beta \leq 2$ under the optimal conditions, with the only restriction given by existence of the limit variance. Moreover, in the range $\beta < 2$, we are able to find the optimal rate of the convergence. It is known that for $\beta = 2$, this rate is at least exponential. On the other had, we establish that for arbitrary $\beta < 2$ the optimal rate is $1/R$.

Our work is inspired by a series of papers due to Borodin, Okounkov [5] and Tracy, Widom [4], who first suggested the algebro-combinatorial approach to the circular unitary ensemble (the case $\beta = 2$), and Jiang, Matsumoto, who extended it to arbitrary temperature. In particular, we generalize Gessel's expansion [10] of expectations of multiplicative functionals under the circular unitary ensemble via the Schur polynomials. For arbitrary β, we express these expectations via the Jack polynomials. Our result implies the analogue of the Szegó theorem for the circular β-ensemble, first proved by Johansson [7] via the loop equations. In [3], Lambert conjectures that the optimal condition of 1/2-Sobolev regularity of the function is sufficient if and only if $\beta \leq 2$. Our work verifies that sufficiency. These results for the circular β-ensemble admit taking the infinite-particle microscopic limit, which allows us to analyze the sine-β process.

References

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